THE ESTABLISHED MOTION OF A VISCOPLASTIC MEDIUM BETWEEN TWO COAXIAL CONES AND IN A SECTOR BOUNDED BY TWO SIDES

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Usually the properties of viscoplastic media are studied using a cylindrical viscosimeter. There is an error in similar measurements because of the effect of the bottom. Below we consider the flow of a medium in a conical rotating viscosimeter which does not have this disadvantage. We obtain equations for the physical characteristics of the material even if the viscous properties of the material are nonlinear.

In the second problem we consider the flow of a viscoplastic medium inside a sector bounded by two sides, the boundaries of which are rotating slowly and monotonically. It is proved that there are no rigid domains in this flow. This result can be used in certain industrial processes. Thus, in the pouring of a viscoplastic polymer into a mold it is frequently necessary that the polymer should occupy the whole volume of the mold. This is difficult to achieve if the mold has recesses. As the polymer fills the mold, a rigid domain is formed near an edge, and this acts as a "bottleneck" and prevents the mass flowing into the corner. If the boundary of the mold can be made movable, the bottleneck is broken. To a large extent this helps to fill the region at the bottom of the corner with the mass.

In both problems the motion is assumed to be quasistationary, and the effect of temperature and inertia forces is ignored.

1. A rheological model of an incompressible viscoplastic medium is shown in Fig. 1, where the elements of the viscosity V and the plasticity P are combined in parallel. Let s_{ij}^V and s_{ij}^P denote the stress tensors of these elements. We can write the condition for plasticity with yield point k_1 in the Mises form:

$$s_{ij}^p\,(s_{ij}^p - {}^1/_3 s_{kk}\delta_{ij}) = 2k_1^2 \tag{1.1}$$

The associative law of the flow yields an equation defining the element P in the form [1]

$$s_{ij}^p = \sqrt{2}k_1 I_2^{-1}\varepsilon_{ij} + {}^1/_3 s_{kk}^p\delta_{ij}, \quad I_2^2 = \varepsilon_{ke}\varepsilon_{ke} \geqslant 0, \quad \varepsilon_{ij} = {}^1/_2\,(v_{i,j} + v_{j,i}) \tag{1.2}$$

where v_i is the velocity of the particles of the medium. We can write the equation for an element of the viscosity in the form

$$s_{ij}^v = 2\eta\,(I_2)\,\varepsilon_{ij} + {}^1/_3 s_{kk}^v\delta_{ij} \tag{1.3}$$

The relation between the viscosity coefficient η and the invariant I_2 must be such that the second law of thermodynamics holds

$$s_{ij}^v\varepsilon_{ij} = 2I_2^2\eta\,(I_2) \geqslant 0, \quad \text{or} \quad \eta\,(I_2) \geqslant 0 \tag{1.4}$$

In addition to this inequality, the viscosity coefficient must satisfy the equation

$$\lim I_2^2\,\eta\,(I_2) = 0 \qquad (I_2 \to 0) \tag{1.5}$$

This condition implies that the rate of dissipation of energy in the medium due to its viscous properties tends to zero as the rate of deformation tends to zero.

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The stresses σ_{ij} in the continuous medium are the sums of the stresses on the elements of viscosity and plasticity. From this we obtain the equation defining the viscoplastic medium with nonlinear viscosity

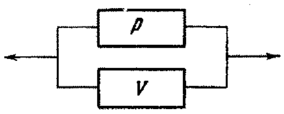

Fig. 1

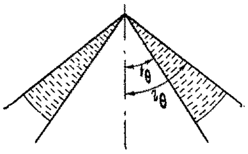

Fig. 2

$$\delta_{ij} = [\sqrt{2}k_1 I_2^{-1} + 2\eta(I_2)]\,\varepsilon_{ij} - p\delta_{ij} \tag{1.6}$$

where p is the hydrostatic pressure.

2. Let us consider the flow of a viscoplastic medium between two coaxial cones (Fig. 2). The medium fills the space between the cones. The free surface of the medium is assumed to be spherical with radius r_0. The inner core, with semiapex angle θ_1, rotates with constant angular velocity ω_0 and the outer cone, with semiapex angle θ_2, does not move. This model is called a conical viscosimeter. We shall discuss the problem in a spherical coordinate system (r, φ, Ψ).

We assume that the velocity field has the form

$$v_1 = v_2 = 0 \qquad v_3 = r\omega(\varphi)\sin\varphi \tag{2.1}$$

This implies that every particle moves in a circle of radius r sin φ about the axis of the cones with constant angular velocity $\omega(\varphi)$. There is only one nonzero component of the rate of deformation

$$\varepsilon_{23} = \frac{1}{2}\frac{d\omega}{d\varphi}\sin\varphi = \frac{1}{2}\gamma, \qquad I_2 = \frac{1}{\sqrt{2}}|\gamma| \tag{2.2}$$

The sign of γ depends on the direction of rotation of the inner cone; hence, for the sake of definiteness, we shall assume that γ is of positive sign. We can write the equations of equilibrium in the spherical coordinate system in the following form for the given problem:

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial \varphi} = 0, \quad \frac{d\tau}{d\varphi} + 2\tau\,\mathrm{ctg}\,\varphi = 0, \quad \tau = \sigma_{23} \tag{2.3}$$

If we integrate these equations, we obtain

$$p = c_0, \qquad \tau = c_1 / \sin^2\varphi \tag{2.4}$$

From (1.6), (2.2), and (2.4), we derive the differential equation

$$\eta(\gamma)\gamma + k = \frac{c_1}{\sin^2\varphi}, \quad \text{or} \quad F(\omega'\sin\varphi) = \frac{c_1}{\sin^2\varphi} - k \tag{2.5}$$

$$F(\gamma) = \eta(\gamma)\gamma, \qquad k = k_1\,\mathrm{sign}\,\gamma \tag{2.6}$$

The constant of integration c_0 is determined after specifying the pressure at some point in the domain of the flow. The quantity c_1 and the other constant of integration of (2.5) are determined from the two boundary conditions

$$\omega = \omega_0 \ \text{for} \ \varphi = \theta_1, \qquad \omega = 0 \ \text{for} \ \varphi = \theta_2 \tag{2.7}$$

or

$$\omega = \omega_0 \ \text{for} \ \gamma = \theta_1, \qquad \omega = \omega' = 0 \ \text{for} \ \varphi = \theta^* \tag{2.8}$$

The boundary conditions (2.7) are used if the whole medium between the cones is involved in the motion, while (2.8) is used if only part of the medium adjacent to the inner core takes part in the motion. In the latter case we must have

$$\theta^* \leqslant \theta_2 \tag{2.9}$$

We assume that the yield point and the relation between the viscosity coefficient and the rate of deformation are unknown and have to be determined. We can show that this can be done using an experiment with a conical viscosimeter. We note that the determination of the function $\eta(\gamma)$ is equivalent to the determination of the function $F(\gamma)$.

From observation on the motion of particles of the medium at the free surface we can find the form of the equation

$$\omega = \omega(\varphi) \tag{2.10}$$

Knowing $\omega(\varphi)$, we can easily determine the functions

$$\gamma = \gamma(\varphi), \qquad \varphi = \varphi(\gamma) \tag{2.11}$$

Substituting $\varphi = \varphi(\gamma)$ into the right side of (2.8), we find

$$F(\gamma) = \frac{c_1}{\sin^2\varphi(\gamma)} - k \tag{2.12}$$

It remains to determine k and c_1. We assume that part of the domain occupied by the medium adjacent to the inner core is at rest. This is possible if ω_0 is small. At the free surface of the medium the boundary of the region involved in the motion is well observed and the angle θ^* is determined. At the surface of the cone with semiapex angle θ^* bounding the region of motion of the medium, the rate of deformation vanishes and so from (2.5) we obtain

$$c_1 = k\sin^2\theta^* \tag{2.13}$$

We find the yield point k as follows. If ω_0 is very small, but not zero, the rate of deformation is also small. In this case at the surface of the inner cone the stress is at the yield point. This allows us to express the yield point in terms of the torional moment M_0 which must be applied to the inner cone in order to move it from its position

$$k = 3M_0 / 2\pi r_0^3 \sin^2\theta_1 \tag{2.14}$$

Substituting (2.13) and (2.14) into (2.12), we can obtain equations for $F(\gamma)$ and $\eta(\gamma)$.

If we already know the form of the function $F(\gamma)$, for example,

$$F(\gamma) = \eta_0\gamma^m \tag{2.15}$$

the problem of determining the nonlinear relation between the viscosity coefficient and the rate of deformation is considerably simplified. In the case of (2.15) this problem can be reduced to the determination of the coefficient η_0 and the exponent m.

To determine η_0 and m we substitute (2.15) into (2.5). As a result we obtain the differential equation

$$\frac{d\omega}{d\varphi}\sin\varphi = \left(\frac{c_1}{\eta_0\sin^2\varphi} - \frac{k}{\eta_0}\right)^{1/m}$$

From this we have

$$\omega(\varphi) = \int_{\theta_1}^{\varphi} \frac{1}{\sin\varphi}\left(\frac{c_1}{\eta_0\sin^2\varphi} - \frac{k}{\eta_0}\right)^{1/m} d\varphi + \omega_0 \tag{2.16}$$

This integral can be computed explicitly if $1/m$ is an integer. If we are using the boundary conditions (2.8) the constant c_1 is found from (2.13) and the angle θ^* must satisfy the equation

$$\int_{\theta_1}^{\theta^*} \frac{1}{\sin\varphi}\left(\frac{c_1}{\eta_0\sin^2\varphi} - \frac{k}{\eta_0}\right)^{1/m} d\varphi + \omega_0 = 0 \tag{2.17}$$

After computing θ^* from (2.17), we find η_0 from

$$\eta_0^{1/m} = -\frac{1}{\omega_0}\int_{\theta_1}^{\theta^*} \frac{1}{\sin\varphi}\left(\frac{c_1}{\sin^2\varphi} - k\right)^{1/m} d\varphi \tag{2.18}$$

The exponent m is determined if we compute the angular velocity ω at some intermediate point between the cones and use (2.16).

If ω_0 is large, so that (2.9) does not hold, then, by (2.7), c_1 is determined by the equation

$$\int_{\theta_1}^{\theta_2} \frac{1}{\sin\varphi}\left(\frac{c_1}{\eta_0\sin^2\varphi} - \frac{k}{\eta_0}\right)^{1/m} d\varphi + \omega_0 = 0 \tag{2.19}$$

For a linear viscosity law (m = 1), from (2.16) we find

$$\omega = \frac{c_1 - 2k}{2\eta_0} \ln\left(\operatorname{tg}\frac{\varphi}{2} \Big/ \operatorname{tg}\frac{\theta_1}{2}\right) - \frac{c_1}{2\eta_0}\left(\frac{\cos\varphi}{\sin^2\varphi} - \frac{\cos\theta_1}{\sin^2\theta_1}\right) + \omega_0 \tag{2.20}$$

3. Suppose a medium filling the open sector bounded by two sides with angle 2θ moves with velocity ω due to a slow monotonic change in the angle θ. There is no flow of the medium in the direction of the edge of the sector. We shall discuss this problem in a cylindrical coordinate system $x_1 = r$, $x_2 = \varphi$, $x_3 = z$.

The incompressibility condition is

$$\frac{\partial v_1}{\partial r} + \frac{v_1}{r} + \frac{1}{r}\frac{\partial v_2}{\partial \varphi} = 0 \tag{3.1}$$

It follows from (3.1) that there is a potential $\Phi(r, \varphi)$ such that

$$v_1 = \frac{1}{r}\frac{\partial \Phi}{\partial \varphi}, \qquad v_2 = -\frac{\partial \Phi}{\partial r} \tag{3.2}$$

We seek the solution of the problem in the form

$$\Phi(r, \varphi) = r^2 f(\varphi), \quad v_1 = rf', \quad v_2 = -2rf \tag{3.3}$$

The nonzero components of the rate of deformation are

$$\varepsilon_{11} = -\varepsilon_{22} = f', \qquad \varepsilon_{12} = {}^1/_2 f'' \tag{3.4}$$

For the nonzero stress components we have, from (1.6) and (3.4),

$$\sigma_{11} = 2\beta f' + p(r, \varphi), \quad \sigma_{22} = -2\beta f' + p, \quad \sigma_{12} = \beta f''$$
$$2\beta = \sqrt{2}kI_2^{-1} + 2\eta(I_2) \tag{3.5}$$

In (3.5) only p depends on the two coordinates r and φ; the remaining variables depend only on φ. We can write the function $p(r, \varphi)$ in the form

$$p = p_0 \ln r + p_1(\varphi) \tag{3.6}$$

We have to substitute (3.5) and (3.6) in the equilibrium equation

$$\frac{\partial \sigma_{11}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{12}}{\partial \varphi} + \frac{\sigma_{11} - \sigma_{22}}{r} = 0, \quad \frac{\partial \sigma_{12}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{22}}{\partial \varphi} + \frac{2\sigma_{12}}{r} = 0 \tag{3.7}$$

After substituting (3.5) into (3.7) we obtain two differential equations for $p_1(\varphi)$ and $f(\varphi)$

$$p_1' = 2\beta' f', \quad 2I_2 p_0 + (f''' + 4f')\,[2I_2\beta + (f'')^2 d\beta / dI_2] = 0 \tag{3.8}$$

Because the medium adheres to the moving boundaries, the solution of equations (3.8) has to satisfy the four boundary conditions

$$f' = 0, \quad f = \pm\omega \quad \text{for} \quad \varphi = \pm\theta \tag{3.9}$$

If the moving boundaries are slippery, then, from the absence of slip and the condition that the medium does not flow across the boundary we find the boundary conditions

$$f'' = 0, \quad f = \pm\omega \quad \text{for} \quad \varphi = \pm\theta \tag{3.10}$$

In both problems the flow of the medium is symmetric about the bisector of the angle. From the symmetry of the flow of the medium and the expression (3.3) for the velocity it follows that the function $f(\varphi)$ is antisymmetric. In this connection the boundary conditions (3.9) and (3.10) can be simplified and reduced to the following forms

$$f' = 0, \quad f = \omega \quad \text{for} \quad \varphi = \theta \tag{3.11}$$
$$f'' = 0, \quad f = \omega \quad \text{for} \quad \varphi = \theta \tag{3.12}$$

The general antisymmetric solution $f(\varphi)$ of the first equation of (3.8) depends on one arbitrary constant of integration and the unknown constant p_0, which are determined from the two boundary conditions (3.11) or (3.12). Obviously

$$f = f'' = 0 \quad \text{for} \quad \varphi = 0 \tag{3.13}$$

This implies that the bisector of the angle can be assumed to be a fixed boundary at which there is no adhesion. Thus, the solution of the problem with the boundary conditions (3.11) is at the same time the solution of the problem of the flow of a medium in a sector bounded by two sides forming an angle θ when there is no slipping at the moving boundary. If we consider (3.12) and (3.13) together, we find that the solution of the problem with the boundary conditions (3.12) is also the solution of the similar problem of the flow of a medium in a sector bounded by two sides forming an angle θ in which one boundary is movable and the other fixed. By analogy with (3.13) we must have $f'' = 0$ for $\varphi = \frac{1}{2}\theta$ on the bisector of the angle θ. We can continue to bisect the angle, and each time we must have $f'' = 0$ on the bisector of each sector. This leads to the conclusion that the solution of the problem with the boundary conditions (3.12) must satisfy the equation

$$f'' = 0, \qquad f = (\omega \,/\, \theta)\, \varphi \tag{3.14}$$

Substituting (3.14) into (3.8), we find that

$$\begin{aligned} p_0 &= -2k \operatorname{sign} \omega - 4\,(\omega \,/\, \theta)\, \eta\, (\sqrt{2}|\omega/\theta)\\ p_1(\varphi) &= \text{const}, \qquad \sigma_{kk} = 3\,(p_0 \ln r + \text{const}) \end{aligned} \tag{3.15}$$

The constant is determined by specifying the pressure at some point of the domain of the flow. As we saw from (3.14), the velocity field does not depend on the viscous and plastic properties of the medium. These properties only make a contribution to the distribution of the stress field

$$\sigma_{11} - p = p - \sigma_{22} = k \operatorname{sign} \omega + 2\eta\,(\sqrt{2}|\omega/\theta|), \quad \sigma_{12} = 0 \tag{3.16}$$

We shall prove that, in the flow of a medium with velocity field (3.3) satisfying (3.8), there is no rigid kernel. If there is such a kernel, at its boundary the following equations must be satisfied:

$$\begin{aligned} v_x &= r\,(f'\cos\varphi + 2f\sin\varphi) = u_0\\ v_y &= r\,(f'\sin\varphi - 2f\cos\varphi) = 0, \qquad \varepsilon_{ij} = 0 \end{aligned} \tag{3.17}$$

where u_0 is the velocity of motion of the rigid kernel parallel to the axis of symmetry – the x axis. Since u_0 is independent of r and φ, from (3.17) we find the conditions for the supposed boundary of the rigid kernel

$$f = f' = f'' = 0 \tag{3.18}$$

We can show that (3.18) implies that all subsequent derivatives of f vanish. In approaching the boundary of the rigid kernel from the side of the domain of the flow, we find in the limit that

$$\lim\left(\frac{I_2}{f''}\right)^2 = \frac{1}{2} + 2\lim\left(\frac{f'}{f''}\right)^2 = \frac{1}{2} + 2\lim\left(\frac{f''}{f'''}\right)^2 \tag{3.19}$$

If $f''' \neq 0$ at the boundary of the rigid kernel, the limit in (3.19) is $\frac{1}{2}$. The expression in brackets in the second equation of (3.8) is nonzero and so $f''' = 0$. Differentiating (3.8) successively several times, we find that all the derivatives of f vanish. Hence the solution must be trivial, but it does not satisfy the boundary conditions (3.11) and (3.12). This contradiction proves that there is no rigid kernel.

The general solution of equation (3.8) with the boundary conditions (3.11) has not been obtained. For the special cases when $\theta = \theta_1 = \frac{1}{4}\pi$ or $\theta_2 = \frac{3}{4}\pi$, if we assume that I_2 = const, we find the solution of the problem (3.8), (3.11) in the form

$$\begin{aligned} f &= a \sin 2\varphi, \quad a = (-1)^{i+1} \quad (i = 1,2)\\ I_2 &= 2\sqrt{2}|a|, \quad \beta = k/4|a| + \eta\,(I_2), \quad p_0 = 0, \quad p_1 = \text{const}\\ \sigma_{11} - p_1 &= p_1 - \sigma_{22} = 4\beta a \cos 2\varphi, \qquad \sigma_{12} = -4\beta a \sin 2\varphi \end{aligned} \tag{3.20}$$

For a viscous fluid, when k = 0 and $\eta = \eta_0$ = const, the solution of equation (3.8) with the boundary conditions (3.11) has the form

$$f = \frac{\omega\,(2\varphi\cos 2\theta - \sin 2\varphi)}{2\theta\cos 2\theta - \sin 2\theta}, \quad p = \frac{8\eta_0\omega\cos 2\theta}{\sin 2\theta - 2\theta\cos 2\theta}\ln r + \text{const} \tag{3.21}$$

In the flow of a viscous incompressible fluid under consideration the velocity field is independent of the viscosity coefficient. As $\theta \to 0$ the stresses and the velocity grow unboundedly. There is a critical value of the angle θ^* which is the least positive root of the transcendental equation $2\theta = \tan 2\theta$. As $\theta \to \theta^* - 0$, we must have $\omega > 0$, so that $p > 0$ in the neighborhood of the origin. When $\theta \to \theta^* + 0$, then, if we are to have $p > 0$, we must have $\omega < 0$. If these conditions do not hold, from (3.21) we find that for any choice of of the constant there can be found an r_0 such that for $r < r_0$ the pressure becomes negative. In the regions where the pressure becomes negative there are cavitational bubbles. In this connection the solution of (3.21) can only be used in the neighborhood of the origin defined by the inequality $r \ll r_0$.

As the boundaries of the sector close ($\theta \to 0$), the stresses become infinite. But we can choose the function $\omega(\theta)$ so that the stresses are finite, when $\theta \to 0$. From (3.21) for small θ we have

$$p = 3\eta_0\theta^{-3}\omega(\theta) \ln r + \text{const} \tag{3.22}$$

From (3.22) we find that for finite p we must have

$$d\theta / dt = \omega(\theta) = -\alpha\theta^3 \tag{3.23}$$

where α is a coefficient of proportionality. Integrating (3.23), we obtain

$$\theta = (\theta_0^{-2} + 2\alpha t)^{-1/2} \tag{3.24}$$

It follows from (3.24) that if the stresses are to remain finite it takes an infinite length of time to close the boundaries of the sector.

The problem of the flow of a viscoplastic medium in a sector parallel to its edge was considered in [2].

LITERATURE CITED

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